In mechanics, the quantity

$$A = \int_{t_1}^{t_2} \sum_i p_i \dot{q}_i \, dt$$ is defined as action.

The principle of least action for conservative system is then expressed as

$$\Delta \int_{t_1}^{t_2} \sum p_i \dot{q}_i \, dt = 0$$

where Δ is the variation.

<u>Proof:</u> we have

$$A = \int_{t_1}^{t_2} \sum p_i \dot{q}_i \, dt \quad \text{——— (1)}$$

$$= \int_{t_1}^{t_2} (L + H) \, dt$$

$$= \int_{t_1}^{t_2} L dt + \int_{t_1} H dt$$

$$= \int_{t_1}^{t_2} L dt + H(t_2 - t_1) \quad \text{——— (2)}$$

Since H is conserved, Δ variation of action is

$$\Delta A = \Delta \int_{t_1}^{t_2} L dt + H \Delta (t_2 - t_1)$$

$$= \Delta \int_{t_1}^{t_2} L dt + H \Delta t \Big|_{t_1}^{t_2} \quad \text{——— (3)}$$

Now solving the integral,

$$\Delta \int_{t_1}^{t_2} L dt$$

Since t_1 and t_2 limits are also subject to change in this variation, Δ cannot be taken inside the integral.

Let
$$\int_{t_1}^{t_2} L\,dt = I \qquad \text{so that} \qquad \dot{I} = L$$

Since $\Delta \equiv \delta + \Delta t \frac{d}{dt}$, using this we get

$$\Delta \int_{t_1}^{t_2} L\,dt = \Delta I(t_2) - \Delta I(t_1)$$
$$= [\delta I(t_2) + \dot{I}(t_2)\Delta(t_2)] - [\delta I(t_1) + \dot{I}(t_1)(\Delta t_1)]$$
$$= [\delta I(t_2) - \delta I(t_1) + \dot{I}(t_2)\Delta(t_2) - \dot{I}(t_1)\Delta t_1]$$
$$= \delta I \Big|_{t_1}^{t_2} + L\Delta(t_2) - L\Delta(t_1)$$

$$\Delta \int_{t_1}^{t_2} L\,dt = \delta \int_{t_1}^{t_2} L\,dt + L\Delta t \Big|_{t_1}^{t_2} \qquad \text{—— (4)}$$

Substituting equation (4) in (3), we get

$$\Delta A = \delta \int_{t_1}^{t_2} L\,dt + L\Delta t \Big|_{t_1}^{t_2} + H\Delta t \Big|_{t_1}^{t_2} \qquad \text{—— (5)}$$

Since $\delta \int_{t_1}^{t_2} L\,dt$ cannot be zero in consequence of Hamilton's principle.

Hamilton principle requires that $\delta q_i = 0$ at the end points of the path, but in this variation $\Delta q_i = 0$ at the end points and not δq_i. Therefore, the integral will not vanish. Using the nature of δ-variation, the integral can be expressed as

$$\delta\int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i\right)dt$$

$$= \int_{t_1}^{t_2} \sum_i \left[\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}(\delta q_i)\right]dt$$

[where $\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)$; from Lagrange's eqn. of motion]

Thus,

$$\delta\int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} \sum_i \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right)\right]dt$$

Putting $\delta q_i = \Delta q_i - \dot{q}_i \Delta t$, we get

$$\delta\int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} \sum_i \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\Delta q_i - \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\Delta t\right)\right]dt$$

$$= \sum_i \left(\frac{\partial L}{\partial \dot{q}_i}\Delta q_i - \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\Delta t\right]_{t_1}^{t_2}$$

At end points $\Delta q_i = 0$, Therefore

$$\delta\int_{t_1}^{t_2} L\,dt = -\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\Delta t\Big|_{t_1}^{t_2}$$

$$= -\sum_i p_i \dot{q}_i \Delta t\Big|_{t_1}^{t_2}$$

With this substitution eqn. (5) becomes

$$\Delta A = -\sum_i p_i\dot{q}_i\Delta t\Big|_{t_1}^{t_2} + L\Delta t\Big|_{t_1}^{t_2} + H\Delta t\Big|_{t_1}^{t_2}$$

$$= \left(H + L - \sum p_i \dot{q}_i\right)\Delta t\Big|_{t_1}^{t_2}$$

$$= 0$$

Since $H = \sum_i p_i \dot{q}_i - L$

Thus, it proves that

$$\Delta \int_{t_1}^{t_2} \sum p_i \dot{q}_i \, dt = 0$$

Which is the principle of least action.

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